

ON THE RANDOM VARIABLE

$$\mathbb{N}^r \ni (k_1, k_2, \dots, k_r) \mapsto \gcd(n, k_1 k_2 \cdots k_r) \in \mathbb{N}$$

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1. INTRODUCTION AND MAIN THEOREM

In his talk at JAMI 2009 on March 24, Kurokawa presented a rather mysterious looking identity of elementary number theory:

$$(1) \quad \frac{1}{n} \sum_{k=1}^n \gcd(n, k) = \prod_{p|n} \left(1 + \left(1 - \frac{1}{p} \right) \text{ord}_p(n) \right)$$

Example 1.1. $n = 12 = 2^2 \cdot 3$:

$$\begin{aligned} \frac{1}{12}(1 + 2 + 3 + 4 + 1 + 6 + 1 + 4 + 3 + 2 + 1 + 12) &= \frac{40}{12} = \frac{10}{3} \\ &= 2 \cdot \frac{5}{3} = \left(1 + \left(1 - \frac{1}{2} \right) \cdot 2 \right) \cdot \left(1 + \left(1 - \frac{1}{3} \right) \cdot 1 \right) \end{aligned}$$

In fact, this is a special case (the case $r = 1$) of identities obtained by [KO]:

Theorem of Kurokawa-Ochiai [KO]. *For $n, r \in \mathbb{N}$,*

$$(2) \quad \frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r) = \prod_{p|n} \left[\sum_{l=0}^r \text{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right]$$

Here, ${}_m H_r$ is the repeated combination:

$${}_m H_r := \# \{ \text{degree } r \text{ homogeneous monomials in } m \text{ variables} \}$$

$$= {}_{r+(m-1)} C_{m-1} = {}_{m+r-1} C_r = (-1)^r \binom{-m}{r}$$

$$c.f. \quad (1+x)^m = \sum_{r=0}^m {}_m C_r x^r, \quad (1-x)^{-m} = \sum_{|x|<1} \sum_{r=0}^{\infty} \binom{-m}{r} (-x)^r = \sum_{r=0}^{\infty} {}_m H_r x^r$$

Example 1.2. $n = 6 = 2 \cdot 3$, $r = 2$:

$$\begin{aligned} \frac{1}{6^2} & (1 + 2 + 3 + 2 + 1 + 6 + 2 + 2 + 6 + 2 + 2 + 6 + 3 + 6 + 3 + 6 + 3 + 6 \\ & + 2 + 2 + 6 + 2 + 2 + 6 + 1 + 2 + 3 + 2 + 1 + 6 + 6 + 6 + 6 + 6 + 6) \\ &= \frac{133}{36} = \frac{7}{4} \cdot \frac{19}{9} = \frac{4+2+1}{4} \cdot \frac{9+6+4}{9} \\ &= \left(1 + 1 \cdot \left(1 - \frac{1}{2} \right) + 1 \cdot \left(1 - \frac{1}{2} \right)^2 \right) \cdot \left(1 + 1 \cdot \left(1 - \frac{1}{3} \right) + 1 \cdot \left(1 - \frac{1}{3} \right)^2 \right) \end{aligned}$$

[KO] obtained (2) by studying some multivariable zeta function of Igusa type, and Kurokawa said he is not aware of any elementary proof even for (1).

Now the purpose of this paper is to give a purely elementary proof of generalizations of Kurokawa-Ochiai identities (2) from the view point of elementary probability theory. Fixing $n \in \mathbb{N}$, we would like to understand the random variable:

$$(3) \quad \begin{aligned} \tilde{X} : \tilde{\Omega} &:= \mathbb{N}^r \rightarrow \mathbb{N} \\ (k_1, k_2, \dots, k_r) &\mapsto \gcd(n, k_1 k_2 \cdots k_r), \end{aligned}$$

Although $\tilde{\Omega} = \mathbb{N}^r$ is an infinite set, we would like to regard it being equipped with the “homogeneous measure”. For this purpose, we observe:

$$k_i \equiv k'_i \pmod{n} \quad (i = 1, 2, \dots, r) \implies \gcd(n, k_1 k_2 \cdots k_r) = \gcd(n, k'_1 k'_2 \cdots k'_r)$$

Thus, instead of (3), we may equally consider the following random variable:

$$(4) \quad \begin{aligned} X : \Omega &:= \{1, 2, \dots, n\}^r \rightarrow \mathbb{N} \\ (k_1, k_2, \dots, k_r) &\mapsto \gcd(n, k_1 k_2 \cdots k_r), \end{aligned}$$

where $\Omega = \{1, 2, \dots, n\}^r$ is equipped with the homogeneous measure. Then the Kurokawa-Ochiai identity (2) is nothing but a convenient formula to evaluate the average $E[X]$ of the random variable (4). Now, from the view point of elementary probability theory, it is very natural to seek for similar convenient formulae for the variance $V[X] = E[X^2] - E[X]^2$ and even “higher” invariants.

Our Main Theorem offers such formulae for the continuous version $E[X^w]$ ($w \in \mathbb{C}$). Their special cases $w \in \mathbb{N}$ are nothing but the moments of the random variable X , and the simplest case $w = 1$ is nothing but the Kurokawa-Ochiai identity (2).

Main Theorem. For $n, r \in \mathbb{N}$, $w \in \mathbb{C}$,

$$\begin{aligned} & \frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r)^w \\ &= \begin{cases} \prod_{p|n} \left[\left(\frac{1-p^{-1}}{1-p^{w-1}} \right)^r + p^{\text{ord}_p(n)(w-1)} \sum_{l=0}^{r-1} \text{ord}_p(n) H_l \left\{ (1-p^{-1})^l - \left(\frac{1-p^{-1}}{1-p^{w-1}} \right)^r (1-p^{w-1})^l \right\} \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left[\sum_{l=0}^r \text{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right] & \text{if } w = 1 \end{cases} \\ &= \begin{cases} \prod_{p|n} \left[\left(\frac{1-p^{-1}}{1-p^{w-1}} \right)^r + p^{\text{ord}_p(n)(w-1)} (1-p^{-1})^r \sum_{l=0}^{r-1} \text{ord}_p(n) H_l \left\{ (1-p^{-1})^{l-r} - (1-p^{w-1})^{l-r} \right\} \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left[\sum_{l=0}^r \text{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right] & \text{if } w = 1 \end{cases} \end{aligned}$$

Corollary 1.3. For $n \in \mathbb{N}$ and $w \in \mathbb{C}$ satisfying $w = 1$ or $|p^w - 1| > 1$,

$$(5) \quad \lim_{r \rightarrow \infty} \frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r)^w = n^w$$

Heuristically speaking, (5) implies the random variable (3) “converges” to the constant random variable concentrated at n .

When n is fixed, the number of the terms needed to be evaluated in $\frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r)^w$ grow exponentially with respect to r , but Main Theorem allows us to evaluate it with linear growth with respect to r .

In our purely elementary proof for Main Theorem, the original identity (1) turns out to be a matter of triviality. In the course of proving the general case, we shall

show some identities involving the repeated combinations (Lemma 2.1), which may be of independent interest.

We note another kind of generalizations of (1) is given in [DKK], which is once again proved using some zeta function of Igusa type. Even for this and some generalization, we can offer an elementary proof [M1]. It is possible to define and study some multivariable *deformed* zeta function of \mathbb{F}_1 -scheme of Igusa-type [M2], which generalizes both [DKK] and [KO].

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2. PROOF OF MAIN THEOREM

Whereas [KO] used some multivariable Igusa type zeta functions for groups, we exploit some ring structure: For a finite ring with n elements $R = \{k_i \in R \mid 1 \leq i \leq n\}$ and $r \in \mathbb{N}$, set

$$Z_R^r(w) := \frac{1}{|R|^r} \sum_{(k_1, \dots, k_r) \in R^r} |R/(k_1 \cdots k_r)R|^w \quad (w \in \mathbb{C})$$

We wish to understand this, because

$$Z_{\mathbb{Z}/n\mathbb{Z}}^r(w) = \frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r)^w \quad (w \in \mathbb{C})$$

Of course, we have an elementary probability theoretical interpretation: For the random variable

$$X : \Omega := R^r \rightarrow \mathbb{N} \\ (k_1, k_2, \dots, k_r) \mapsto |R/(k_1 \cdots k_r)R|,$$

where $\Omega = R^r$ is equipped with the homogeneous measure,

$$E[X^w] = \frac{1}{|R|^r} \sum_{(k_1, \dots, k_r) \in R^r} |R/(k_1 \cdots k_r)R|^w = Z_R^r(w)$$

Elementary properties of $Z_R^r(w)$:

- If there is a finite ring decomposition $R = \prod_i R_i$, then

$$Z_R^r(w) = \prod_i Z_{R_i}^r(w)$$

- Set $N_R^r(f) := \left| \{(k_1, \dots, k_r) \in R^r \mid |R/(k_1 \cdots k_r)R| = f\} \right|$, then

$$Z_R^r(w) := \frac{1}{|R|^r} \sum_{(k_1, \dots, k_r) \in R^r} |R/(k_1 \cdots k_r)R|^w = \frac{1}{|R|^r} \sum_{f=1}^{|R|} N_R^r(f) f^w$$

So, if $n = \prod_{p|n} p^{\text{ord}_p(n)}$, then by the Chinese Remainder Theorem,

$$(6) \quad \begin{aligned} \frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r)^w &= Z_{\mathbb{Z}/n\mathbb{Z}}^r(w) = \prod_{p|n} Z_{\mathbb{Z}/p^{\text{ord}_p(n)}\mathbb{Z}}^r(w) \\ &= \prod_{p|n} \frac{1}{p^{r \text{ord}_p(n)}} \sum_{f=1}^{p^{\text{ord}_p(n)}} N_{\mathbb{Z}/p^e\mathbb{Z}}^r(f) f^w \end{aligned}$$

Thus, we need to compute

$$\begin{aligned} N_{\mathbb{Z}/p^e\mathbb{Z}}^r(f) &= \left| \left\{ (k_1, \dots, k_r) \in (\mathbb{Z}/p^e\mathbb{Z})^r \mid |(\mathbb{Z}/p^e\mathbb{Z})/(k_1 \cdots k_r)(\mathbb{Z}/p^e\mathbb{Z})| = f \right\} \right| \\ &= \left| \left\{ (k_1, \dots, k_r) \in \{1, 2, \dots, p^e\}^r \mid \gcd(p^e, k_1 \cdots k_r) = f \right\} \right| \end{aligned}$$

for $1 \leq f \leq p^{\text{ord}_p(n)}$.

To get some feeling, we first play with the simplest case $r = 1$. Then, it is easy to evaluate:

$$(7) \quad N_{\mathbb{Z}/p^e\mathbb{Z}}^1(f) = \begin{cases} p^{e-d} - p^{e-d-1} & \text{if } f = p^d \ (0 \leq d < e) \\ 1 & \text{if } f = p^e \\ 0 & \text{if otherwise} \end{cases}$$

which follows from

$$\# \{1, 2, \dots, p^e\} = p^e = \left(\sum_{d=0}^{e-1} \underbrace{(p^{e-d} - p^{e-d-1})}_{\text{ord}_p=d} \right) + \underbrace{p^{e-e}}_{\text{ord}_p=e}$$

Proof for $r = 1$:

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \gcd(n, k)^w &\stackrel{(6)}{=} \prod_{p|n} \frac{1}{p^{\text{ord}_p(n)}} \sum_{f=1}^{p^{\text{ord}_p(n)}} N_{\mathbb{Z}/p^e\mathbb{Z}}^1(f) f^w \\ &\stackrel{(7)}{=} \prod_{p|n} \frac{1}{p^{\text{ord}_p(n)}} \left[\left\{ \sum_{d=0}^{\text{ord}_p(n)-1} (p^{\text{ord}_p(n)-d} - p^{\text{ord}_p(n)-d-1}) (p^d)^w \right\} + \left\{ 1 \cdot (p^{\text{ord}_p(n)})^w \right\} \right] \\ &= \prod_{p|n} \left[\left\{ (1 - p^{-1}) \sum_{d=0}^{\text{ord}_p(n)-1} (p^{w-1})^d \right\} + p^{\text{ord}_p(n)(w-1)} \right] \\ &= \begin{cases} \prod_{p|n} \left((1 - p^{-1}) \frac{1 - p^{(w-1)\text{ord}_p(n)}}{1 - p^{w-1}} + p^{\text{ord}_p(n)(w-1)} \right) & \text{if } w \neq 1 \\ \prod_{p|n} \left(1 + \left(1 - \frac{1}{p} \right) \text{ord}_p(n) \right) & \text{if } w = 1 \end{cases} \\ &= \begin{cases} \prod_{p|n} \left[\left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right) + p^{\text{ord}_p(n)(w-1)} \left(1 - \frac{1 - p^{-1}}{1 - p^{w-1}} \right) \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left(1 + \left(1 - \frac{1}{p} \right) \text{ord}_p(n) \right) & \text{if } w = 1 \end{cases} \end{aligned}$$

□

So, the case $r = 1$ is a matter of triviality, and not much extra effort in considering general $w \in \mathbb{C}$. However, the proof for general $r \in \mathbb{N}$ is more complicated.

Observe for $0 \leq d < e$,

$$\begin{aligned}
 N_{\mathbb{Z}/p^e\mathbb{Z}}^r(p^d) &= \text{The coefficient of } x^d \text{ in} \\
 &\quad \{(p^e - p^{e-1}) \cdot 1 + (p^{e-1} - p^{e-2}) \cdot x + \dots + (p-1)x^{e-1}\}^r \\
 &= (p^e - p^{e-1})^r \times \text{The coefficient of } x^d \text{ in } \left\{1 + \left(\frac{x}{p}\right) + \dots + \left(\frac{x}{p}\right)^{e-1}\right\}^r \\
 (8) \quad &= (p^e - p^{e-1})^r \times \text{The coefficient of } x^d \text{ in } \left\{1 - \left(\frac{x}{p}\right)\right\}^{-r} \\
 &= (p^e - p^{e-1})^r \times \text{The coefficient of } x^d \text{ in } \sum_{l=0}^{e-1} {}_rH_l \left(\frac{x}{p}\right)^l \\
 &= (p^e - p^{e-1})^r {}_rH_d \left(\frac{1}{p}\right)^d
 \end{aligned}$$

To evaluate $N_{\mathbb{Z}/p^e\mathbb{Z}}^r(p^e)$ and for further computation, we prove the following identities involving the repeated combinations, which may be of independent interest:

Lemma 2.1. *For $e, r \in \mathbb{N}$, set a degree $e-1$ polynomial of x by*

$$(9) \quad f_e^r(x) := \sum_{k=0}^{e-1} {}_rH_k x^k$$

Then,

$$(10a) \quad f_e^r(1) = {}_{r+1}H_{e-1} = {}_{e+r-1}C_{e-1} = {}_{e+r-1}C_r = {}_eH_r$$

$$(10b) \quad (1-x)^r f_e^r(y) + y^e f_r^e(1-x) = (1-x)^r \left(f_e^r(y) - y^e f_e^r(1)\right) + y^e f_{r+1}^e(1-x)$$

$$(10c) \quad (1-x)^r f_e^r(x) + x^e f_r^e(1-x) = 1$$

Proof. (10a) is an easy consequence of the Taylor expansions at $x = 0$:

$$\begin{aligned}
 f_e^r(1) &= \sum_{k=0}^{e-1} {}_rH_k = \text{The coefficient of } x^{e-1} \text{ in } \left(\sum_{k=0}^{e-1} {}_rH_k x^k\right) \left(\sum_{k=0}^{e-1} x^k\right) \\
 &= \text{The coefficient of } x^{e-1} \text{ in } \left((1-x)^{-r}(1-x)^{-1} = (1-x)^{-(r+1)}\right) \\
 &= {}_{r+1}H_{e-1} = {}_{e+r-1}C_{e-1} = {}_{e+r-1}C_r = {}_eH_r
 \end{aligned}$$

Now (10b) is an immediate consequence of (10a):

$$\begin{aligned}
 (1-x)^r f_e^r(y) + y^e f_r^e(1-x) &= (1-x)^r \left(f_e^r(y) - y^e f_e^r(1)\right) + y^e \left(f_r^e(1-x) + f_e^r(1)(1-x)^r\right) \\
 \stackrel{(9)(10a)}{=} &(1-x)^r \left(f_e^r(y) - y^e f_e^r(1)\right) + y^e \left(\left(\sum_{k=0}^{r-1} {}_eH_k (1-x)^k\right) + {}_eH_r (1-x)^r\right) \\
 &= (1-x)^r \left(f_e^r(y) - y^e f_e^r(1)\right) + y^e \sum_{k=0}^r {}_eH_k (1-x)^k \stackrel{(9)}{=} (1-x)^r \left(f_e^r(y) - y^e f_e^r(1)\right) + y^e f_{r+1}^e(1-x)
 \end{aligned}$$

To prove (10c), consider the Taylor expansions of degree e of $(1-x)^{-r}$ at $x=0$, and degree r of $x^{-e} = (1-(1-x))^{-e}$ at $x=1$, respectively,

$$(11a) \quad f_e^r(x) = \sum_{k=0}^{e-1} {}_rH_k x^k = (1-x)^{-r} + x^e \cdot O(1; x \rightarrow 0)$$

$$(11b) \quad f_r^e(1-x) = \sum_{k=0}^{r-1} {}_eH_k (1-x)^k = x^{-e} + (1-x)^r \cdot O(1; x \rightarrow 1)$$

We now set the left hand side polynomial of x in (9) to be

$$(12) \quad l(x) := (1-x)^r f_e^r(x) + x^e f_r^e(1-x)$$

Then, applying (11a) and (11b) respectively to (12), we see

$$(13) \quad \begin{aligned} l(x) &= (1-x)^r \left((1-x)^{-r} + x^e \cdot O(1; x \rightarrow 0) \right) + x^e f_r^e(1-x) \\ &= 1 + x^e \cdot \left((1-x)^r O(1; x \rightarrow 0) + f_r^e(1-x) \right) \end{aligned}$$

$$(14) \quad \begin{aligned} l(x) &= (1-x)^r f_e^r(x) + x^e \left(x^{-e} + (1-x)^r \cdot O(1; x \rightarrow 1) \right) \\ &= 1 + (1-x)^r \cdot \left(f_e^r(x) + x^e \cdot O(1; x \rightarrow 1) \right) \end{aligned}$$

(13) and (14) respectively imply $l(x) - 1$ is divisible by x^e and $(1-x)^r$. Thus, $l(x) - 1$, a polynomial of x of degree at most $e+r-1$, is divisible by $x^e(1-x)^r$. Of course, this implies $l(x) = 1$, and thus (10c) has been proven. \square

Now we may complete our computation of $N_{\mathbb{Z}/p^e\mathbb{Z}}^r(f)$:

Proposition 2.2. *For the cyclic ring $\mathbb{Z}/p^e\mathbb{Z}$ with p a prime and $e \in \mathbb{N}$,*

(15)

$$N_{\mathbb{Z}/p^e\mathbb{Z}}^r(f) = \begin{cases} (p^e - p^{e-1})^r {}_rH_d \left(\frac{1}{p} \right)^d & \text{if } f = p^d \ (0 \leq d < e) \\ p^{e(r-1)} f_r^e \left(1 - \frac{1}{p} \right) = p^{e(r-1)} \sum_{l=0}^{r-1} {}_eH_l \left(1 - \frac{1}{p} \right)^l & \text{if } f = p^e \\ 0 & \text{if otherwise} \end{cases}$$

Proof of Proposition 2.2. It is obvious that $N_{\mathbb{Z}/p^e\mathbb{Z}}^r(f) = 0$ if f is not of the form p^d ($0 \leq d \leq e$). The case of $0 \leq d < e$ is already treated in (8).

Finally, the case of $d = e$ can be taken care of as follows:

$$\begin{aligned} N_{\mathbb{Z}/p^e\mathbb{Z}}^r(p^e) &= \left| (\mathbb{Z}/p^e\mathbb{Z})^r \right| - \sum_{d=0}^{e-1} N_{\mathbb{Z}/p^e\mathbb{Z}}^r(p^d) \stackrel{(8)}{=} (p^e)^r - \sum_{d=0}^{e-1} (p^e - p^{e-1})^r {}_rH_d \left(\frac{1}{p} \right)^d \\ &\stackrel{(9)}{=} (p^e)^r - (p^e - p^{e-1})^r f_e^r \left(\frac{1}{p} \right) = p^{er} \left(1 - \left(1 - \frac{1}{p} \right)^r f_e^r \left(\frac{1}{p} \right) \right) \\ &\stackrel{(10c)}{=} p^{er} \left(\frac{1}{p} \right)^e f_r^e \left(1 - \frac{1}{p} \right) = p^{e(r-1)} f_r^e \left(1 - \frac{1}{p} \right) = p^{e(r-1)} \sum_{l=0}^{r-1} {}_eH_l \left(1 - \frac{1}{p} \right)^l \end{aligned}$$

\square

Proof of Main Theorem.

$$\begin{aligned}
& \frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \dots k_r)^w \stackrel{(6)}{=} \prod_{p|n} \frac{1}{p^{r \operatorname{ord}_p(n)}} \sum_{f=1}^{p^{\operatorname{ord}_p(n)}} N_{\mathbb{Z}/p^e \mathbb{Z}}^r(f) f^w \\
& \stackrel{(15)}{=} \prod_{p|n} \frac{1}{p^{r \operatorname{ord}_p(n)}} \left[\left\{ \sum_{d=0}^{\operatorname{ord}_p(n)-1} (p^{\operatorname{ord}_p(n)} - p^{\operatorname{ord}_p(n)-1})^r {}_r H_d \left(\frac{1}{p} \right)^d (p^d)^w \right\} \right. \\
& \quad \left. + \left\{ p^{\operatorname{ord}_p(n)(r-1)} \sum_{l=0}^{r-1} \operatorname{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right\} (p^{\operatorname{ord}_p(n)})^w \right] \\
& = \prod_{p|n} \left[\left\{ (1 - p^{-1})^r \sum_{d=0}^{\operatorname{ord}_p(n)-1} {}_r H_d (p^{w-1})^d \right\} + \left\{ p^{\operatorname{ord}_p(n)(w-1)} \sum_{l=0}^{r-1} \operatorname{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right\} \right] \\
& \stackrel{(9)}{=} \prod_{p|n} \left[(1 - p^{-1})^r f_{\operatorname{ord}_p(n)}^r(p^{w-1}) + (p^{w-1})^{\operatorname{ord}_p(n)} f_r^{\operatorname{ord}_p(n)}(1 - p^{-1}) \right] \\
& \stackrel{(10b)}{=} \begin{cases} \prod_{p|n} \left[(1 - p^{-1})^r f_{\operatorname{ord}_p(n)}^r(p^{w-1}) + (p^{w-1})^{\operatorname{ord}_p(n)} f_r^{\operatorname{ord}_p(n)}(1 - p^{-1}) \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left[(1 - p^{-1})^r \left(f_{\operatorname{ord}_p(n)}^r(p^{w-1}) - (p^{w-1})^{\operatorname{ord}_p(n)} f_{\operatorname{ord}_p(n)}^r(1) \right) + (p^{w-1})^{\operatorname{ord}_p(n)} f_{r+1}^{\operatorname{ord}_p(n)}(1 - p^{-1}) \right] & \text{if } w = 1 \end{cases} \\
& \stackrel{(10c)}{=} \begin{cases} \prod_{p|n} \left[(1 - p^{-1})^r \frac{1 - (p^{w-1})^{\operatorname{ord}_p(n)} f_r^{\operatorname{ord}_p(n)}(1 - p^{-1})}{(1 - p^{w-1})^r} + (p^{w-1})^{\operatorname{ord}_p(n)} f_r^{\operatorname{ord}_p(n)}(1 - p^{-1}) \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left[f_{r+1}^{\operatorname{ord}_p(n)}(1 - p^{-1}) \right] & \text{if } w = 1 \end{cases} \\
& \stackrel{(9)}{=} \begin{cases} \prod_{p|n} \left[\left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right)^r + p^{\operatorname{ord}_p(n)(w-1)} \sum_{l=0}^{r-1} \operatorname{ord}_p(n) H_l \left\{ (1 - p^{-1})^l - \left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right)^r (1 - p^{w-1})^l \right\} \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left[\sum_{l=0}^r \operatorname{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right] & \text{if } w = 1 \end{cases} \\
& = \begin{cases} \prod_{p|n} \left[\left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right)^r + p^{\operatorname{ord}_p(n)(w-1)} (1 - p^{-1})^r \sum_{l=0}^{r-1} \operatorname{ord}_p(n) H_l \left\{ (1 - p^{-1})^{l-r} - (1 - p^{w-1})^{l-r} \right\} \right] & \text{if } w \neq 1 \\ \prod_{p|n} \left[\sum_{l=0}^r \operatorname{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right] & \text{if } w = 1 \end{cases}
\end{aligned}$$

□

Proof of Corollary 1.3. When $w = 1$,

$$\begin{aligned}
& \lim_{r \rightarrow +\infty} \prod_{p|n} \left[\sum_{l=0}^r \operatorname{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right] = \prod_{p|n} \left[\sum_{l=0}^{+\infty} \operatorname{ord}_p(n) H_l \left(1 - \frac{1}{p} \right)^l \right] \\
& \stackrel{|1 - \frac{1}{p}| < 1}{=} \prod_{p|n} \left(1 - \left(1 - \frac{1}{p} \right) \right)^{-\operatorname{ord}_p(n)} = \prod_{p|n} p^{\operatorname{ord}_p(n)} = n
\end{aligned}$$

When $|p^w - 1| > 1$, since $|1 - p^{-1}| < 1 < |1 - p^w|$,

$$\begin{aligned}
& \lim_{r \rightarrow +\infty} \prod_{p|n} \left[\left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right)^r + p^{\text{ord}_p(n)(w-1)} \sum_{l=0}^{r-1} \text{ord}_p(n) H_l \left\{ (1 - p^{-1})^l - \left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right)^r (1 - p^{w-1})^l \right\} \right] \\
&= \prod_{p|n} \left[\lim_{r \rightarrow +\infty} \left(\frac{1 - p^{-1}}{1 - p^{w-1}} \right)^r + p^{\text{ord}_p(n)(w-1)} \sum_{l=0}^{+\infty} \text{ord}_p(n) H_l (1 - p^{-1})^l \right. \\
&\quad \left. - p^{\text{ord}_p(n)(w-1)} \lim_{r \rightarrow +\infty} (1 - p^{-1})^r \sum_{l=0}^{r-1} \text{ord}_p(n) H_l (1 - p^{w-1})^{l-r} \right] \\
&\stackrel{| \frac{1-p^{-1}}{1-p^{w-1}} | < 1, |1-p^{-1}| < 1}{=} \prod_{p|n} \left[0 + p^{\text{ord}_p(n)(w-1)} p^{\text{ord}_p(n)} \right. \\
&\quad \left. - p^{\text{ord}_p(n)(w-1)} \lim_{r \rightarrow +\infty} (1 - p^{-1})^r \sum_{l=0}^{r-1} \text{ord}_p(n) H_l (1 - p^{w-1})^{l-r} \right] \\
&= \prod_{p|n} \left[p^{\text{ord}_p(n)(w)} - p^{\text{ord}_p(n)(w-1)} \lim_{r \rightarrow +\infty} (1 - p^{-1})^r \sum_{l=0}^{r-1} \text{ord}_p(n) H_l (1 - p^{w-1})^{l-r} \right]
\end{aligned}$$

This is shown to be equal to n^w , if the following is shown:

$$\lim_{r \rightarrow +\infty} (1 - p^{-1})^r \sum_{l=0}^{r-1} \text{ord}_p(n) H_l (1 - p^{w-1})^{l-r} = 0$$

However, this follows from

$$\begin{aligned}
& \lim_{r \rightarrow +\infty} \left| (1 - p^{-1})^r \sum_{l=0}^{r-1} \text{ord}_p(n) H_l (1 - p^{w-1})^{l-r} \right| = \lim_{r \rightarrow +\infty} \frac{\left| \sum_{l=0}^{r-1} \text{ord}_p(n) H_l \left(\frac{1}{1 - p^{w-1}} \right)^{r-l} \right|}{\left| (1 - p^{-1})^{-1} \right|^r} \\
&\leq \lim_{r \rightarrow +\infty} \frac{\sum_{l=0}^{r-1} \text{ord}_p(n) H_l \left| \left(\frac{1}{1 - p^{w-1}} \right)^{r-l} \right|}{\left| (1 - p^{-1})^{-1} \right|^r} \stackrel{|p^w - 1| > 1}{\leq} \lim_{r \rightarrow +\infty} \frac{\sum_{l=0}^{r-1} \text{ord}_p(n) H_l \cdot 1}{\left| (1 - p^{-1})^{-1} \right|^r} \\
&\stackrel{(9)}{=} \lim_{r \rightarrow +\infty} \frac{f_r^{\text{ord}_p(n)}(1)}{\left| (1 - p^{-1})^{-1} \right|^r} \stackrel{(10a)}{=} \lim_{r \rightarrow +\infty} \frac{\text{ord}_p(n) + r - 1}{\left| (1 - p^{-1})^{-1} \right|^r} C_{\text{ord}_p(n)} = 0,
\end{aligned}$$

where the last equality follows since $\text{ord}_p(n) + r - 1 C_{\text{ord}_p(n)}$ is a degree $\text{ord}_p(n)$ polynomial of r and $\left| (1 - p^{-1})^{-1} \right| > 1$. Now the claim has been proven. \square

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